REGULARITY AND RELAXED PROBLEMS OF MINIMIZING BIHARMONIC MAPS INTO SPHERES

Min-Chun Hong and Changyou Wang

Abstract. For $n \geq 5$ and $k \geq 4$, we show that any minimizing biharmonic map from $\Omega \subset \mathbb{R}^n$ to S^k is smooth off a closed set whose Hausdorff dimension is at most $n-5$. When $n=5$ and $k=4$, for a parameter $\lambda \in [0,1]$ we introduce a λ-relaxed energy $\mathbb{H}_\lambda$ of the Hessian energy for maps in $W^{2,2}(\Omega; S^4)$ so that each minimizer u_λ of $\mathbb{H}_\lambda$ is also a biharmonic map. We also establish the existence and partial regularity of a minimizer of $\mathbb{H}_\lambda$ for $\lambda \in [0,1)$.

1. Introduction

For $n \geq 5$ and $k \geq 4$, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $S^k \subset \mathbb{R}^{k+1}$ be the unit sphere. Define

$$W^{2,2}(\Omega, S^k) = \{u \in W^{2,2}(\Omega, \mathbb{R}^{k+1})| \quad |u(x)| = 1 \text{ for a.e. } x \in \Omega\}$$

and also define, for a given map $u_0 \in W^{2,2}(\Omega, S^k)$,

$$W^{2,2}_{u_0}(\Omega, S^k) = \{u \in W^{2,2}(\Omega, S^k)| \quad u - u_0|_{\partial\Omega} = \nabla(u-u_0)|_{\partial\Omega} = 0 \text{ in the trace sense }\}.$$

The hessian energy functional on $W^{2,2}(\Omega, S^k)$ is defined by

$$\mathbb{H}(u) = \int_\Omega |\triangle u|^2\, dx, \quad \forall u \in W^{2,2}(\Omega, S^k). \tag{1.1}$$

Recall that a map $u \in W^{2,2}(\Omega, S^k)$ is a (weakly) biharmonic map if u is a critical point of $\mathbb{H}(\cdot)$ in $W^{2,2}(\Omega, S^k)$ so that it satisfies the Euler-Lagrange equation:

$$-\triangle^2 u = (|\triangle u|^2 + 2\nabla\cdot(\nabla u \cdot \triangle u) - \triangle|\nabla u|^2)u \tag{1.2}$$

in the distribution sense, where $\nabla\cdot$ is the divergence operator in $\mathbb{R}^n$ and $\cdot$ is the inner product in $\mathbb{R}^{k+1}$.

A typical class of biharmonic maps is given by minimizing biharmonic maps. A map $u \in W^{2,2}(\Omega, S^k)$ is a minimizing biharmonic map if it satisfies

$$\mathbb{H}(u) \leq \mathbb{H}(w), \quad \forall w \in W_u^{2,2}(\Omega, S^k). \tag{1.3}$$

The study of minimizing biharmonic maps into spheres was initiated by Hardt-Mou [HM]. Chang-Wang-Yang [CWY] established the partial regularity for weakly stationary biharmonic maps into spheres, ie. if $u \in W^{2,2}(\Omega; S^k)$ be a weakly stationary biharmonic map, then $u \in C^\infty(\Omega, S^k)$ for $n = 4$ and $u \in C^\infty(\Omega \setminus \Sigma, S^k)$ with $\mathcal{H}^{n-4}(\Sigma) = 0$ for $n \geq 5$. Very recently, the main theorems of [CWY] have been generalized by the second author [W1,2,3] for stationary biharmonic maps into any compact smooth Riemannian submanifold N of the Euclidean spaces. The results in [CWY], [W1,2,3] give the partial regularity theorem for minimizing biharmonic maps for $n \geq 5$ since a minimizing biharmonic map is stationary. However the minimality of the biharmonic map $\Phi(x, y) = \frac{x}{|x|} : B^5 \times B^{n-5} \to S^4$ (see Proposition A1 of §5 below) indicates that the dimension of the singular set Σ for minimizing biharmonic maps may be smaller than $n - 4$. Compared with the optimal partial regularity for minimzing harmonic maps by Giaquinta-Giusti [GG] and by Schoen-Uhlenbeck [SU], it is natural to ask the following question:

Does the singular set of a minimizing biharmonic map have Hausdorff dimension at most $n - 5$?

In this aspect, we have

Theorem A. *For $n \geq 5$ and $k \geq 4$, let $u \in W^{2,2}(\Omega, S^k)$ be a minimizing biharmonic map and denote by $\mathcal{S}(u)$ the singular set of u. Then $\mathcal{S}(u)$ is discrete for $n = 5$, and has its Hausdorff dimension at most $n - 5$ for $n \geq 6$.*

One of key ingredients to prove Theorem A is to derive an extension inequality (2.1) for the hessian energy for maps into S^k with $k \geq 4$, inspired by Hardt-Lin's extension Lemma (see [HL], [HKL]) in the context of harmonic maps into a simply connected manifold. An important consequence of (2.1) is to obtain the Caccioppoli inequality (2.7) for mininimizing biharmonic maps so that *a weakly convergent sequence of minimizing biharmonic maps in $W^{2,2}(\Omega, S^k)$ converges strongly in $W^{2,2}_{loc}$ to a minimizing biharmonic map.* Combined this fact with the energy monotonicity inequality for stationary biharmonic maps due to [CWY], it guarantees that a refinement of Federer dimension reduction scheme [F] is applicable so that Theorem A follows from a similar argument as one in [S].

For $n = 5$, it follows from Theorem A that suitable rescalings at each singular point of a minimizing biharmonic map $u \in W^{2,2}(\Omega, S^4)$ yields a minimizing biharmonic map of the form $\Psi(\frac{x}{|x|})$ for some $\Psi \in C^\infty(S^4, S^4)$. Motivated by the work by Brezis-Coron-Lieb [BCL] on minimizing harmonic maps from B^3 into S^2, it will be an interesting problem to study the map Ψ given as above. Inspired by the problem proposed by Hardt-Lin

[HL1] on the context of harmonic maps from B^3 to S^2, we would also like to ask the following question:

For any given map $\psi \in C^\infty(S^4, S^4)$ with zero degree, is there a biharmonic map $u \in C^\infty(\overline{B^5}, S^4)$ with $u = \psi$ on ∂B^5?

The example A2 of Section 5 indicates that any minimizing biharmonic map extension of some boundary map with degree zero has singularities. In order to study this problem, we extend the idea of a relaxation of the Dirichlet energy functional of harmonic maps from B^3 to S^2 by Bethuel-Brezis-Coron [BBC] and Giaquinta-Modica-Soucèk [GMS]. More precisely, we hope to introduce a relaxed energy functional for biharmonic maps from $\Omega \subset \mathbb{R}^5$ to S^4.

For a $W^{2.2}$-map from $\Omega \subset \mathbb{R}^5$ to S^4, the D-field of u, $D(u) = (D_1(u), ..., D_5(u)) \in L^1(\Omega, \mathbb{R}^5)$, is defined by

$$D_1(u) = \det\left(u, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_5}\right), \quad ..., \quad D_5(u) = \det\left(u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_4}\right).$$

For any given $\phi \in W^{2,2}(\Omega, S^4) \cap C^\infty(\partial\Omega, S^4)$, define

$$L(u) := \frac{1}{\sigma_4} \sup_{\xi:\Omega\to\mathbb{R}, \|\nabla\xi\|_{L^\infty}\le 1} \{\int_\Omega D(u)\cdot\nabla\xi\, dx - \int_{\partial\Omega} D(u)\cdot\nu\xi\, dH^{n-1}\},\ \forall u \in W^{2,2}_\phi(\Omega, S^4).$$

For any $\lambda \in (0, 1]$, the λ-relaxed hessian energy functional is defined by

$$\mathbb{H}_\lambda(u) := \mathbb{H}(u) + 16\lambda\sigma_4 L(u),\ \forall u \in W^{2,2}_\phi(\Omega, S^4). \tag{1.3}$$

Throughout the paper, we denote by $\sigma_k = \mathcal{H}^k(S^k)$ the area of the unit sphere $S^k \subset \mathbb{R}^{k+1}$ for $k \geq 4$.

Then we have

Theorem B.

(a) For any $\lambda \in (0, 1]$, $\mathbb{H}_\lambda$ is sequentially lower semi-continuous in $W^{2,2}_\phi(\Omega, S^4)$ for the weak $W^{2,2}$-topology,

(b) for any $\lambda \in (0, 1]$, there exists a $u_\lambda \in W^{2,2}_\phi(\Omega, S^4)$ which minimizes $\mathbb{H}_\lambda(\cdot)$ among $W^{2,2}_\phi(\Omega, S^4)$-maps, and

(c)for any $\lambda \in (0, 1)$, u_λ is a weakly biharmonic map satisfying $u_\lambda \in C^\infty(\Omega \setminus \Sigma_\lambda, S^4)$, with $\mathcal{H}^{1-\delta}(\Sigma_\lambda) = 0$ for some $\delta > 0$.

Finally, modifying the arguments of [BCL], there exist infinitely many weak biharmonic maps in $W^{2,2}_{x/|x|}(\Omega; S^4)$. It will be an interesting question to establish this result

for general boundary data. To do it, one needs to establish the boundary regularity of a minimizing harmonic maps, but this is unknown. The partial regularity has been established in [LW].

The paper is organized as follows. In Section 2, we derive a Caccioppoli's inequality for Q-minimizing biharmonic maps. In Section 3, we prove a partial regularity for Q-minimizing biharmonic maps and also present a proof of Theorem A. In Section 4, we prove Theorems B. In Section 5 is an appendix and several elementary facts will be given.

Acknowledgment. We would like to thank Professor Mariano Giaquinta for useful comments and discussions. The research of the first author is supported by the Australian Research Council.

2. Caccioppoli's inequality

In this section, we consider, for any $Q \geq 1$, Q-minimizing biharmonic maps from $\mathbb{R}^n$ to S^k for $n \geq 5$ and $k \geq 4$, and establish the Cacciopoli inequality. The idea is inspired by the Hardt-Lin's extension Lemma (see [HL], [HKL]).

Definition 2.0. *(Q-minimizing biharmonic map) Let Q be a constant with $1 \leq Q < \infty$. For $n \geq 5$ and $k \geq 4$, a map $u \in W^{2,2}(\Omega, S^k)$ is called a Q-minimizing biharmonic map if (i) u is a weakly biharmonic map and (ii) u satisfies*

$$\mathbb{H}(u) \leq Q\mathbb{H}(v), \ \forall v \in W^{2,2}_u(\Omega, S^k). \tag{2.0}$$

It is clear that any minimizing biharmonic map is a Q-minimizing biharmonic map with $Q = 1$. Now we have

Lemma 2.1. *(Extension Lemma) For $n \geq 5$ and $k \geq 4$, let $\Omega \subset \mathbb{R}^n$ be any bounded smooth domain. Then for any map $v \in W^{2,2}(\Omega, \mathbb{R}^{k+1})$ with $|v(x)| = 1$ and $\nabla v(x) \in T_{v(x)}S^k$ for a.e. $x \in \partial\Omega$, there exists a map $w \in W^{2,2}(\Omega, S^k)$ such that $w = v, \nabla w = \nabla v$ on $\partial\Omega$, and*

$$\int_\Omega |\triangle w|^2 \, dx \leq C \int_\Omega (|\triangle v|^2 + |\nabla v|^4) \, dx, \tag{2.1}$$

where $C > 0$ is independent of u and Ω.

Proof. For any $a \in \mathbb{R}^{k+1}$ with $|a| \leq \frac{1}{2}$, consider the map

$$w_a(x) = \frac{v(x) - a}{|v(x) - a|}, \quad x \in \Omega.$$

Then a simple calculation gives

$$\nabla w_a(x) = |v(x)-a|^{-1}\nabla v(x) - |v(x)-a|^{-3}(v(x)-a)\otimes(v(x)-a)\nabla v(x),\ \forall x\in\Omega,$$

we have, by taking one more derivative of this identity,

$$|\triangle w_a(x)| \le C\left(\frac{|\triangle v(x)|}{|v(x)-a|} + \frac{|\nabla v(x)|^2}{|v(x)-a|^2}\right),\ \forall x\in\Omega. \tag{2.2}$$

Integrating (2.2) over $\Omega\times B^{k+1}_{\frac{1}{2}}$ with respect to (x,a) and applying Fubini's theorem, we have

$$\begin{aligned}
&\int_{B^{k+1}_{1/2}}\int_\Omega |\triangle w_a|^2(x)\,dx\,da = \int_\Omega\int_{B^{k+1}_{1/2}} |\triangle w_a|^2(x)\,dx\,da\\
&\le C\int_\Omega \left(|\triangle v|^2 + |\nabla v|^4\right)(x)\cdot\left[\int_{B^{k+1}_{\frac{1}{2}}}\left(\frac{1}{|v(x)-a|^2}+\frac{1}{|v(x)-a|^4}\right)da\right]dx\\
&\le C\int_\Omega\left(|\nabla v|^4 + |\triangle v|^2\right)(x)\,dx,
\end{aligned}$$

where we use the fact that

$$\int_{B^{k+1}_{1/2}}\frac{1}{|v(x)-a|^4}\,da \le c(k) = \begin{cases}\frac{16}{9}\frac{\sigma_k}{k+1}, & \text{for } |v|\ge 1\\ \frac{1}{k-3}(\frac{3}{2})^{k-3}\sigma_k, & \text{for } |v|\le 1.\end{cases}$$

Therefore we can find an $a_0\in B^{k+1}_{\frac{1}{2}}$ such that

$$\int_\Omega |\triangle w_{a_0}|^2(x)\,dx \le C\int_\Omega(|\triangle v|^2 + |\nabla v|^4)(x)\,dx. \tag{2.3}$$

For $a\in B^{k+1}_{\frac{1}{2}}$, define

$$\Pi_a(\xi) = \frac{\xi-a}{|\xi-a|} : S^k\to S^k.$$

It is easy to see that Π_a is a C^2 diffeomorphism of S^k onto itself. In fact,

$$\Pi_a^{-1}(\xi) = a + [(a\cdot\xi)^2 + (1-|a|^2)]^{1/2}\xi,\ \forall\xi\in S^k.$$

In particular, we have

$$\max_{a\in B^{k+1}_{\frac{1}{2}}}\left(\|\nabla\Pi_a^{-1}\|_{C^0(S^k)}+\|\nabla^2\Pi_a^{-1}\|_{C^0(S^k)}\right)=\Lambda<\infty.$$

Now we set

$$w(x)=\Pi_{a_0}^{-1}\circ w_{a_0}(x)=\Pi_{a_0}^{-1}\circ\Pi_{a_0}(v(x)),\ ,\forall x\in\Omega.$$

Note

$$\begin{aligned}\nabla w(x)&=\nabla\Pi_{a_0}^{-1}(w_{a_0}(x))\nabla w_{a_0}(x),\\ \nabla^2 w(x)&=\nabla\Pi_{a_0}^{-1}(w_{a_0}(x))\nabla^2 w_{a_0}(x)+\nabla^2\Pi_{a_0}^{-1}(w_{a_0}(x))(\nabla w_{a_0}(x),\nabla w_{a_0}(x))\end{aligned}$$

and

$$|\nabla w_{a_0}(x)|^2\le|\triangle w_{a_0}(x)|,\ \text{for a.e. } x\in\Omega$$

due to the fact that $|w_{a_0}(x)|=1$ for a.e. $x\in\Omega$. Then we have

$$|\triangle w|(x)\le C(\Lambda)[|\triangle w_{a_0}|+|\nabla w_{a_0}|^2](x)\le C(\Lambda)|\triangle w_{a_0}|(x),\ \forall x\in\Omega. \tag{2.4}$$

Combined this with (2.3), it implies that $w\in W^{2,2}(\Omega,S^k)$ and satisfies (2.1). To see that w has the same trace as v on $\partial\Omega$, observe that, since $\Pi_{a_0}^{-1}\circ\Pi_{a_0}|_{S^k}$ is the identity map, $w=v$ on $\partial\Omega$. Moreover, since for any $x\in\partial\Omega$ we have $\nabla w(x)=\nabla(\Pi_{a_0}^{-1}\circ\Pi_{a_0})(v(x))(\nabla v(x))$, $\nabla v(x)\in T_{v(x)}S^k$, and $\nabla(\Pi_{a_0}^{-1}\circ\Pi_{a_0})(v(x)):T_{v(x)}S^k\to T_{v(x)}S^k$ is the identity map, we have $\nabla w=\nabla v$ on $\partial\Omega$. The proof of Lemma 2.1 is complete. □

We follow the iteration method in [G] to get

Lemma 2.2. *For $0\le r_0<r_1<\infty$, let $f:[r_0,r_1]\to(0,\infty)$ be a mesurable function. Suppose that there exist $\theta\in(0,1)$, $A>0$ $B>0$, α, and $\beta>0$ such that for $r_0\le t<s\le r_1$ we have*

$$f(t)\le\theta f(s)+[A(s-t)^{-\alpha}+B(s-t)^{-\beta}]. \tag{2.5}$$

Then for all $r_0\le\rho R\le r_1$ we have

$$f(\rho)\le C[A(R-\rho)^{-\alpha}+B(R-\rho)^{-\beta}], \tag{2.6}$$

where $C=C(\alpha,\beta,\theta)>0$.

Now we have

Lemma 2.3. *(Cacciopoli's inequality) For $1 \le Q < \infty$, $n \ge 5$ and $k \ge 4$, let u be a Q-minimizer of $\mathbb{H}$ in $W^{2,2}_{u_0}(\Omega, S^k)$. Then for all $x_0 \in \Omega$ and all $R <$ dist$(x_0, \partial\Omega)$, we have*

$$\int_{B_{R/2}(x_0)} |\triangle u|^2\,dx \le CR^{-4} \int_{B_R(x_0)} (|u-u_{x_0,R}|^2 + |u-u_{x_0,R}|^4)\,dx \tag{2.7}$$

for some constant $C > 0$, where $u_{x_0,R} = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u\,dx$ is the average of u over $B_R(x_0)$.

Proof. It follows from Lemma 2.1 and the Q-minimality of u that

$$\int_\Omega |\triangle u|^2\,dx \le CQ \int_\Omega \left(|\triangle v|^2 + |\nabla v|^4\right)\,dx \tag{2.8}$$

for any $v \in W^2_u(\Omega; \mathbb{R}^{k+1})$, where C is a positive constant.

For any s, t with $R/2 \le t < s \le \frac{3R}{4}$, let ϕ be a cut-off function in $B_s(x_0)$ such that $0 \le \phi \le 1$ in $B_s(x_0)$, $\phi \equiv 1$ in $B_t(x_0)$, $\phi \equiv 0$ outside $B_s(x_0)$, $|\nabla\phi| \le C(s-t)^{-1}$ and $|\triangle \phi| \le CR^{-2}$, where C is a constant independent of s, t and R. Taking $v(x) = u(x) - \phi[u(x) - u_{x_0,R}]$, we have

$$\nabla v = (1-\phi)\nabla u - \nabla\phi[u(x) - u_{x_0,R}]$$

and

$$\triangle v = (1-\phi)\triangle u - \triangle\phi[u(x) - u_{x_0,R}] - 2\nabla\phi\nabla u.$$

By (2.8), we obtain

(2.9)

$$\begin{aligned}\int_{B_t} |\triangle u|^2\,dx \le C\int_{B_s\setminus B_t}(|\triangle u|^2 + |\nabla u|^4)\,dx + \frac{C}{(s-t)^2}\int_{B_s}|\nabla u|^2\,dx \\ + \frac{C}{(s-t)^4}\int_{B_s}|u-u_{x_0,R}|^2\,dx + \frac{C}{(s-t)^4}\int_{B_s}|u-u_{x_0,R}|^4\,dx\end{aligned}$$

for all s, t with $\frac{R}{2} \le t < s \le R$.

Noticing $|u| = 1$, we have

$$|\nabla u|^2 \le |\triangle u|. \tag{2.10}$$

By the filling hole trick in (2.9), there exists a positive $\theta < 1$ such that

$$\begin{aligned}\int_{B_t} |\triangle u|^2\,dx \le \theta\int_{B_s}|\triangle u|^2\,dx + \frac{C}{(s-t)^2}\int_{B_s}|\nabla u|^2\,dx \\ + \frac{C}{(s-t)^4}\int_{B_s}|u-u_{x_0,R}|^2\,dx + \frac{C}{(s-t)^4}\int_{B_s}|u-u_{x_0,R}|^4\,dx\end{aligned}$$

for $\frac{R}{2} \le t < s \le \frac{3R}{4}$. Then it follows from Lemma 2 to obtain

(2.11)
$$\begin{aligned}\int_{B_{R/2}(x_0)} |\triangle u|^2\,dx \le & CR^{-2}\int_{B_{\frac{3R}{4}}(x_0)} |\nabla u|^2\,dx + R^{-4}\int_{B_R(x_0)} |u-(u)_{x_0,R}|^2\,dx \\ & + R^{-4}\int_{B_R(x_0)} |u-(u)_{x_0,R}|^4\,dx.\end{aligned}$$

Let $\phi \in C_0^\infty(B_R(x_0))$ be a cut-off function with $\phi \equiv 1$ in $B_{\frac{3R}{4}}(x_0)$, $0 \le \phi \le 1$ and $|\nabla\phi \le \frac{C}{R}$. Integrating by parts, we have

$$\int_{B_R(x_0)} \phi^2|\nabla u|^2\,dx = -\int_{B_R(x_0)} \triangle u\cdot(u-u_{x_0,R})\phi^2 - 2\int_{B_R(x_0)} \nabla u\cdot(u-u_{x_0,R})\phi\nabla\phi\,dx.$$

Then

$$\int_{B_{\frac{3R}{4}}(x_0)} |\nabla u|^2\,dx \le \varepsilon R^2\int_{B_R(x_0)} |\triangle u|^2\,dx + \frac{C}{R^2}\int_{B_R(x_0)} |u-u_{x_0,R}|^2\,dx \tag{2.12}$$

for a sufficiently small ε.

By (2.11)-(2.12) with a sufficiently small ε, the claim (2.7) follows from the standard trick (e.g. [S; Lemma 2 of Chapter 2]). □

As a direct consequence of Lemma 2.3, we have the following reverse Hölder inequality for Q-minimizing biharmonic maps.

Proposition 2.4. *Let $1 \le Q < \infty$, $n \ge 5$ and $k \ge 1$. Suppose that u be a Q-minimizing biharmonic map in $W^{2,2}(\Omega; S^k)$. Then there exists an exponent $p > 4$ such that $u \in W^{2,p}_{loc}(\Omega, \mathbb{R}^5)$. Moreover, for all $x_0 \in \Omega$ and $R < dist(x_0, \partial\Omega)$, we have*

$$\left(\fint_{B_{R/2}(x_0)|} (|\triangle u|^2+1)^{p/2}\,dx\right)^{1/p} \le C\left(\fint_{B_R(x_0)} (|\triangle u|^2+1)\,dx\right)^{1/2}, \tag{2.13}$$

for some constant C depending n, k and Q, where we denote by the average integration over $B_R(x_0)$
$$\fint_{B_R(x_0)} f(x)\,dx = \frac{1}{|B_R(x_0)}\int_{B_R(x_0)} f(x)\,dx.$$

Proof. By the Poincare inequality, we have

$$\int_{B_R(x_0)} |u-u_{x_0,R}|^4\,dx \le CR^{4+n(1-\frac{4}{q})}\left(\int_{B_R(x_0)} |\nabla u|^q\right)^{4/q}$$

and

$$\int_{B_R(x_0)} |u - u_{x_0,R}|^2 \, dx \leq C R^{2+n(1-\frac{2}{q})} \left(\int_{B_R(x_0)} |\nabla u|^q \right)^{2/q}$$

for some $q < 4$. Then it follows from (2.7) that

$$\fint_{B_{R/2}(x_0)} |\triangle u|^2 \, dx \leq C \left(\fint_{B_R(x_0)} |\nabla u|^q \, dx \right)^{4/q} + C \left(\fint_{B_R(x_0)} |\nabla u|^q \, dx \right)^{2/q}$$

for some $q < 4$. This implies from (2.7) that

$$\fint_{B_{R/2}(x_0)} \left[|\triangle u|^2 + 1 \right] \, dx \leq C \left(\fint_{B_R(x_0)} \left[|\triangle u|^2 + 1 \right]^{q/4} \, dx \right)^{4/q}$$

for every $B_R(x_0) \subset \Omega$ with $q < 4$. By the reverse Hölder inequality (cf. [G; page 122]), there exists an exponent $p > 2$ such that for every $B_R(x_0) \subset \Omega$

$$\left(\fint_{B_{R/2}(x_0)} \left[|\triangle u|^2 + 1 \right]^{p/2} \, dx \right)^{2/p} \leq C \fint_{B_R(x_0)} \left[|\triangle u|^2 + 1 \right] \, dx.$$

By the standard L^p-local estimate of linear elliptic equations of second order, $u \in W^{2,p}_{loc}(\Omega, \mathbb{R}^5)$. $\square$

3. Proof of Theorem A

In this section, we present a proof of Theorem A. The proof consists of three steps: (i)regularity under the smallness of renormalized hessian energy, (ii)$W^{2,2}$-compactness of the space of minimizing biharmonic maps and (iii) blow-up argument utilizing both the energy monotonicity inequality ([CWY]) and Federer's dimension deduction [F].

Lemma 3.1. *For any $1 \leq Q < \infty$, $n \geq 5$, and $k \geq 4$, there exists an $\epsilon_0 \in (0, 1)$ such that if $u \in W^{2,2}(\Omega, S^k)$ is an Q-minimizing biharmonic map satisfying*

$$(2R)^{4-n} \int_{B_{2R}(x_0)} |\triangle u|^2 \leq \epsilon_0^2, \text{ for } B_{2R}(x_0) \subset \Omega \tag{3.1}$$

then $u \in C^\infty(B_R(x_0), S^k)$ and satisfies

$$\|u\|_{C^l(B_R(x_0))} \leq C(n, k, Q, \epsilon_0, l), \ \forall l \geq 1. \tag{3.2}$$

Proof. It is based on the following decay estimate: there exists an $\theta_0 \in (0, 1)$ such that

$$(\theta_0 R)^{4-n} \int_{B_{\theta_0 R}(x_0)} |\triangle u|^2 \, dx \leq (\frac{1}{2}) R^{4-n} \int_{B_R(x_0)} |\triangle u|^2 \, dx. \tag{3.3}$$

Once (3.3) is established, the regularity of u follows from the standard iterations and suitable applications of Morrey's Lemma [M]. To prove (3.3), we argue by contradiction as follows (see [HL] for similar arguments for harmonic maps). By rescalings, we may assume that $x_0 = 0$ and $R = 1$. Suppose (3.3) is false. Then there exists a sequence of minimizing biharmonic maps $\{u_i\} \subset W^{2,2}(B_1, S^k)$ such that

$$\int_{B_1} |\triangle u_i|^2 \, dx = \epsilon_i^2 \to 0$$

but we have, for any $\theta \in (0,1)$,

$$\theta^{4-n} \int_{B_\theta} |\triangle u_i|^2 \, dx > \frac{1}{2} \int_{B_1} |\triangle u_i|^2 \, dx = \frac{1}{2}\epsilon_i^2. \tag{3.4}$$

Define the blow-up sequence $v_i(x) = \frac{u_i(x) - (u_i)_1}{\epsilon_n} : B_1 \to \mathbb{R}^{k+1}$. It is easy see

$$(v_i)_1 = 0, \quad \int_{B_1} |\triangle v_i|^2 \, dx = 1, \quad \int_{B_1} |\nabla v_i|^4 \, dx \leq 1.$$

Therefore we may assume that $v_i \to v_\infty$ weakly in $W^{2,2}(B_1)$, strongly in $W^{1,2}(B_1)$ and $L^4(B_1)$. Since u_i satisfies (1.2), it is easy to see that v_i satisfies

$$-\triangle^2 v_i = \epsilon_i(|\triangle v_i|^2 + 2\nabla \cdot (\nabla v_i \cdot \triangle v_i) - \triangle |\nabla v_i|^2) u_i, \quad \text{in } B_1. \tag{3.5}$$

Letting i tend to infinity, we see $\int_{B_1} |\triangle v_\infty|^2 \leq 1$ and

$$\triangle^2 v_\infty = 0, \quad \text{in } B_1.$$

Therefore $v_\infty \in C^\infty(B_1, \mathbb{R}^{k+1})$ and satisfies, for any $\theta \in (0, \frac{1}{2})$,

$$\theta^{2-n} \int_{B_\theta} |\nabla v_\infty|^2 \, dx \leq C\theta^2, \quad \theta^{-n} \int_{B_\theta} |v_\infty - (v_\infty)_\theta|^4 \, dx \leq C\theta^4. \tag{3.6}$$

Therefore, for i sufficiently large, we have

$$\theta^{2-n} \int_{B_\theta} |\nabla u_i|^2 \, dx \leq C\theta^2 \epsilon_i^2, \quad \theta^{-n} \int_{B_\theta} |u_i - (u_i)_\theta|^4 \, dx \leq C\theta^4 \epsilon_i^4. \tag{3.7}$$

This, combined with (2.7) of Lemma 2.3, implies that for any $\theta \in (0, \frac{1}{4})$

$$\theta^{4-n} \int_{B_\theta} |\triangle u_i|^2 \, dx \leq C(\theta^2 + \theta^4 \epsilon_i^2)\epsilon_i^2 \leq C\theta^2 \epsilon_i^2.$$

This contradicts with (3.4), if we choose sufficiently small $\theta \in (0, \frac{1}{4})$. $\square$

As a consequence of Lemma 3.1, we have the following partial regularity for Q-minimizing biharmonic maps.

Corollary 3.2. *For any $1 \le Q < \infty$, $n \ge 5$, and $k \ge 4$. Suppose that $u \in W^{2,2}(\Omega, S^k)$ is an Q-minimizing biharmonic map. Then there exist a closed set $\Sigma \subset \Omega$ and an $\delta > 0$ such that $u \in C^\infty(\Omega \setminus \Sigma, S^k)$ and $\mathcal{H}^{n-4-\delta}(\Sigma) = 0$.*

Proof. It follows from Lemma 3.1 that the singular set of u is given by

$$\Sigma = \{x \in \Omega | \ \liminf_{r \to 0} r^{4-n} \int_{B_r(x)} | \triangle u|^2 \, dy \ge \epsilon_0^2\}.$$

By Proposition 2.4, we have that $u \in W^{2,p}_{\text{loc}}(\Omega, S^k)$ for some $p > 2$. In particular, we have

$$\Sigma \subset \Sigma_p = \{x \in \Omega | \ \liminf_{r \to 0} r^{2p-n} \int_{B_r(x)} | \triangle u|^p \, dy \ge \epsilon_1\}$$

for some $\epsilon_1 > 0$. Therefore it is well-known (cf [G]) that $\mathcal{H}^{n-2p}(\Sigma) = 0$. $\Box$

Now we want to prove that a sequence of weakly convergent minimizing biharmonic maps is also strongly convergent. More precisely, we have

Lemma 3.3. *For $n \ge 5$ and $k \ge 4$, let $\{u_i\} \subset W^{2,2}(\Omega, S^k)$ be a sequence of minimizing biharmonic maps such that u_i converges weakly in $W^{2,2}$ to a map $u \in W^{2,2}(\Omega, S^k)$. Then u_i converges to u strongly in $W^{2,2}_{loc}(\Omega, S^k)$. Moreover, $u \in W^{2,2}(\Omega, S^k)$ is also a minimizing biharmonic map.*

Proof. The idea is similar to that of [HL]. First, it follows from Proposition 2.4 that there exists an $p > 2$ such that for any compact subset $K \subset\subset \Omega$

$$\sup_{i \ge 1} \|u_i\|_{W^{2,p}(K)} \le C(p, K) < \infty. \tag{3.8}$$

Therefore, by the Rellich's compactness theorem, we may assume that $u_i \to u$ strongly in $W^{1,4}_{\text{loc}}(\Omega, S^k)$. By localization, it suffices to show that u minimizes $\mathbb{H}$ on B_R and $u_i \to u$ strongly in $W^{2,2}(B_R, S^k)$ for any ball $B_{2R} \subset \Omega$.

Let $v \in W^{2,2}(B_{2R}, S^k)$ be any map such that $v = u$ in $B_{2R} \setminus B_R$. For any small $\delta > 0$, let $\eta_\delta \in C_0^\infty(B_{(1+3\delta)R})$ be such that $0 \le \eta_\delta \le 1$, $\eta_\delta \equiv 1$ in $B_{(1+2\delta)R}$, $|\nabla \eta_\delta| \le \frac{2}{\delta R}$, and $|\nabla^2 \eta_\delta| \le \frac{4}{(\delta R)^2}$.

Consider $v_i(x) = \eta_\delta(x) v(x) + (1 - \eta_\delta(x)) u_i(x) : A_\delta \equiv B_{(1+3\delta)R} \setminus B_{(1+\delta)R} \to \mathbb{R}^{k+1}$. Then it is easy to see $v_i \in W^{2,2}(A_\delta, \mathbb{R}^{k+1})$ satisfies the condition of Lemma 2.1 on A_δ. Therefore Lemma 2.1 implies that $w_i \in W^{2,2}_{v_i}(A_\delta, S^k)$ such that

$$\int_{A_\delta} | \triangle w_i|^2 \, dx \le C \int_{A_\delta} (| \triangle v_i|^2 + |\nabla v_i|^4) \, dx.$$

Let

$$\overline{w}_i(x) = \begin{cases} w_i(x) & \text{for } x \in A_\delta \\ v(x) & \text{for } x \in B_{(1+\delta)R}. \end{cases}$$

Then $\overline{w}_i \in W^{2,2}_{u_i}(B_{(1+3\delta)R}, S^k)$ so that the $\mathbb{H}$-minimality of u_i implies

(3.9)

$$\begin{aligned}\int_{B_{(1+3\delta)R}} |\triangle u_i|^2\,dx &\le \int_{B_{(1+3\delta)R}} |\triangle \overline{w}_i|^2\,dx \\ &= \int_{B_{(1+\delta)R}} |\triangle v|^2\,dx + \int_{A_\delta} |\triangle w_i|^2\,dx \\ &\le \int_{B_{(1+\delta)R}} |\triangle v|^2\,dx + C\int_{A_\delta} (|\triangle v_i|^2 + |\nabla v_i|^4)\,dx.\end{aligned}$$

Direct calculations imply

$$\int_{A_\delta} |\nabla v_i|^4\,dx \le C[(\delta R)^{-4}\int_{A_\delta} |u_i - v|^4\,dx + \int_{A_\delta} (|\nabla(u_i - v)|^4 + |\nabla v|^4)\,dx]$$

This, combined with the fact that $v = u$ on A_δ and $u_i \to u$ strongly in $W^{1,4}$, implies

$$\lim_{i\to\infty} \int_{A_\delta} |\nabla v_i|^4\,dx = \int_{A_\delta} |\nabla v|^4\,dx = o(\delta), \tag{3.10}$$

where $\lim_{\delta\to 0} o(\delta) = 0$. We also have

(3.11)

$$\begin{aligned}\int_{A_\delta} |\triangle v_i|^2\,dx \le\ & C[(\delta R)^{-4}\int_{A_\delta} |u_i - v|^2\,dx + (\delta R)^{-2}\int_{A_\delta} |\nabla(u_i - v)|^2\,dx] \\ &+ C\int_{A_\delta} (|\triangle u_i|^2 + |\triangle v|^2)\,dx.\end{aligned}$$

Therefore, by (3.8), we have

(3.12)

$$\begin{aligned}\lim_{i\to\infty} \int_{A_\delta} |\triangle v_i|^2\,dx &= \lim_{i\to\infty} \int_{A_\delta} |\triangle u_i|^2\,dx + \int_{A_\delta} |\triangle v|^2\,dx \\ &\le \lim_{i\to\infty} (\int_{A_\delta} |\triangle u_i|^p)^{\frac{2}{p}} |A_\delta|^{\frac{1-2}{p}} + \int_{A_\delta} |\triangle v|^2\,dx \\ &= o(\delta).\end{aligned}$$

Putting (3.10), (3.11), and (3.12) into (3.9), we obtain

$$\lim_{i\to\infty} \int_{B_{(1+3\delta)R}} |\triangle u_i|^2\,dx \le \int_{B_{(1+\delta)R}} |\triangle v|^2\,dx + o(\delta). \tag{3.13}$$

Letting $v \equiv u$ and $\delta \to 0$, (3.13) implies $u_i \to u$ strongly in $W^{2,2}(B_R)$. Moreover, by the lower semicontinuity, (3.13) also implies

$$\int_{B_{(1+3\delta)R}} |\triangle u|^2 \, dx \le \int_{B_{(1+\delta)R}} |\triangle v|^2 \, dx + o(\delta)$$

this clearly implies the $\mathbb{H}$-minimality of u on B_R. The proof is complete. $\square$

In order to give a proof of theorem A, we also need to recall the following monotonicity inequality, which was established in [CWY] for stationary biharmonic maps.

Lemma 3.4. *For $n \ge 5$ and a compact Riemannian submanifold $N \subset R^{k+1}$ without boundary. Suppose that $u \in W^{2,2}(\Omega, N)$ is a stationary biharmonic map. Then we have, for any $x \in \Omega$ and $0 < \rho \le r < dist(x, \partial\Omega)$,*

(3.14)
$$\begin{aligned} &r^{4-n} \int_{B_r(x)} |\triangle u|^2 \, dy + r^{3-n} \int_{\partial B_r(x)} [4|\nabla u|^2 - 4|\frac{\partial u}{\partial r}|^2 + r\frac{\partial}{\partial r}(|\nabla u|^2)] \, dy \\ &= \rho^{4-n} \int_{B_\rho(x)} |\triangle u|^2 \, dy + \rho^{3-n} \int_{\partial B_\rho(x)} [4|\nabla u|^2 - 4|\frac{\partial u}{\partial \rho}|^2 + \rho\frac{\partial}{\partial \rho}(|\nabla u|^2)] \, dy \\ &+ 4 \int_{B_r(x)\setminus B_\rho(x)} (\frac{|\nabla((y-x)\cdot\nabla u)|^2}{|y-x|^{n-2}} + (n-2)\frac{|(y-x)\cdot\nabla u|^2}{|y-x|^n}) \, dy. \end{aligned}$$

Now we complete a proof of theorem A.

Proof of Theorem A. First it follows from Lemma 3.1 that the singular set $\mathcal{S}(u)$ is defined by

$$\mathcal{S}(u) = \{x \in \Omega : \Theta^{n-4}(u,x) \equiv \liminf_{r\to 0} r^{4-n} \int_{B_r(x)} |\triangle u|^2 \, dy \ge \epsilon_0^2\}.$$

It follows from Lemma 2.3 and (2.12) that there exists a $C = C(n,k) > 0$ such that

$$\Theta^{n-4}(u,x) \le \limsup_{r\to 0} r^{4-n} \int_{B_r(x)} |\triangle u|^2 \, dy \le C, \ \forall x \in \Omega. \tag{3.15}$$

Since minimizing biharmonic maps are stationary biharmonic maps, Lemma 3.4 implies that for any $x \in \Omega$,

$$\sigma^u(x,r) := r^{4-n} \int_{B_r(x)} |\triangle u|^2 + r^{3-n} \int_{\partial B_r(x)} [4|\nabla u|^2 - 4|\frac{\partial u}{\partial r}|^2 + r\frac{\partial}{\partial r}(|\nabla u|^2)]$$

is monotonicially nondecreasing with respect to $r > 0$ so that

$$\sigma^u(x) \equiv \lim_{r\to 0} \sigma^u(x,r)$$

exists for any $x \in \Omega$. It is easy to see $\sigma^u(x) < +\infty$. To see $\sigma^u(x) > -\infty$, let $r > 0$ be a good slice, i.e.,

$$\begin{aligned}&|r^{3-n}\int_{\partial B_r(x)} r\frac{\partial}{\partial r}(|\nabla u|^2)|\\&\le 2^n((2r)^{4-n}\int_{B_{2r}(x)}|\nabla^2 u|^2)^{\frac12}((2r)^{2-n}\int_{B_{2r}(x)}|\nabla u|^2)^{\frac12} \le C\end{aligned}$$

so that $\sigma^u(x,r) \ge -C$. Therefore, by choosing good slices $r \downarrow 0$, we have $\sigma^u(x) \ge -C > -\infty$.

Next, we have

Claim: *For any $x_0 \in \mathcal{S}(u)$ and $r_i \to 0$ there exists a minimizing biharmonic map $\phi \in W^{2,2}_{loc}(\mathbb{R}^n, S^k)$ of homogeneous of degree zero (i.e. $\phi(x) = \phi(\frac{x}{|x|})$) such that after passing to subsequences $u_i(x) \equiv u(x_0 + r_i x)$ converges to a aminimizing biharmonic map ϕ strongly in $W^{2,2}_{loc}(\mathbb{R}^n, S^k)$.*

To show this claim, it follows from (3.15) that for any $R > 0$, $\{u_i\} \subset W^{2,2}(B_R, S^k)$ is a bounded sequence of minimizing biharmonic maps. Therefore, it follows from Lemma 3.3 that there exist a minimizing biharmonic map $\phi \in W^{2,2}(B_R, S^k)$ such that $u_i \to \phi$ strongly in $W^{2,2}(B_R, S^k)$. To see ϕ is homogeneous of degree zero, note that for any $0 < R_1 < R_2 \le R$

$$\begin{aligned}\sigma^{u_i}(0,R_2) - \sigma^{u_i}(0,R_1) &= \sigma^u(x_0, R_2 r_i) - \sigma^u(x_0, R_1 r_i)\\&\to \sigma^u(x_0) - \sigma^u(x_0) = 0, \text{ as } i \to \infty.\end{aligned}$$

This, combined with (3.14) and the lower semicontinuity, implies

$$\begin{aligned}&\int_{B_{R_2}\setminus B_{R_1}} (\frac{|\nabla(x\cdot\nabla\phi)|^2}{|x|^{n-2}} + (n-2)\frac{|x\cdot\nabla\phi|^2}{|x|^n})\,dy\\&\le \lim_{i\to\infty}\int_{B_{R_2}\setminus B_{R_1}} (\frac{|\nabla(x\cdot\nabla u_i)|^2}{|x|^{n-2}} + (n-2)\frac{|x\cdot\nabla u_i|^2}{|x|^n})\,dy = 0.\end{aligned}$$

Therefore $\frac{\partial\phi}{\partial r} = 0$ for a.e. $x \in B_{R_2} \setminus B_{R_1}$, which yields ϕ is of homogeneous of degree zero.

With the help of Lemma 3.1, 3.3, 3.4, and the above claim, the dimension estimation of $\mathcal{S}(u)$ can be proved by a refinement of the dimension reduction argument of Federer. For details, we refer to [S; Chapter 3]. □

4. Proof of Theorem B

This section is devoted to the proof of Theorem B. One of the crucial parts is to establish the sequentially lower semicontinuity of $\mathbb{H}_\lambda(\cdot)$ in $W^{2,2}_\phi(\Omega, S^4)$. Throughout this section we assume that $n = 5$, $\Omega \subset \mathbb{R}^5$, and $\phi \in W^{2,2}(\Omega, S^4) \cap C^\infty(\partial\Omega, S^4)$.

Let's first recall the wedge product in $\mathbb{R}^5$. For the standard orthonormal base $\{\mathbf{e}_i\}_{i=1}^5$ of $\mathbb{R}^5$, the wedge product of four vectors a, b, c, d in $\mathbb{R}^5$, $a \wedge b \wedge c \wedge d \in \mathbb{R}^5$, is given by

$$(a \wedge b \wedge c \wedge d)_i = \det\, (\mathbf{e}_i, a, b, c, d), 1 \le i \le 5.$$

Now we have

Lemma 4.1. *For every $\lambda \in (0, 1]$, $\mathbb{H}_\lambda(\cdot)$ is s.l.s.c. in $W^{2,2}_\phi(\Omega, S^4)$ for the weak $W^{2,2}$ topology.*

Proof. Since the supremum of sequentially lower semicontinuous functions is still a sequentially lower semicontinuous function, it suffices to prove that for any fixed ξ : $\Omega \to \mathbb{R}$ with $\|\xi\|_{L^\infty} \le 1$ the functional

$$\mathbb{H}_{\lambda,\xi}(u) = \int_\Omega |\triangle u|^2\, dx + 16\lambda \int_\Omega D(u) \cdot \nabla\xi\, dx$$

is sequentially lower semicontinuous in $W^{2,2}_\phi(\Omega, S^4)$ for the weak $W^{2,2}$ topology.

Let $\{u^n\} \subset W^{2,2}_\phi(\Omega, S^4)$ converge to $u \in W^{2,2}_\phi(\Omega, S^4)$ weakly in $W^{2,2}(\Omega, S^4) \cap W^{1,4}(\Omega, S^4)$, and strongly in $W^{1,2}(\Omega, S^4)$. Set $v^n = u^n - u \in W^{2,2}(\Omega, \mathbb{R}^5)$. Then we have

$$\int_\Omega |\triangle u^n|^2\, dx = \int_\Omega |\triangle v^n|^2\, dx + \int_\Omega |\triangle u|^2\, dx + o(1), \tag{4.1}$$

where $o(1)$ is such that $\lim_{n\to\infty} o(1) = 0$.

Now we claim

$$\int_\Omega |\nabla v^n|^4\, dx \le \int_\Omega |\triangle v^n|^2\, dx + o(1). \tag{4.2}$$

To show (4.2), observe that since $|u^n| = 1$ and $|u| = 1$, we have

$$|v^n|^2 = -2v^n \cdot u$$

this implies, by taking two derivatives,

$$|\nabla v^n|^2 = -(\triangle v^n \cdot v^n + \triangle v^n \cdot u + v^n \cdot \triangle u + 2\nabla v^n \cdot \nabla u).$$

Since we have, for a.e. $x \in \Omega$,

$$(\triangle u \cdot v^n) \to 0,\ (\triangle u \cdot v^n)u^n \to 0,\ (\triangle u \cdot v^n)\nabla u \to 0$$

and

$$\max\{|(\triangle u \cdot v^n)|,\ |(\triangle u \cdot v^n)u^n|\} \le 2|\triangle u| \in L^2(\Omega),\ |(\triangle u \cdot v^n)\nabla u| \le |\triangle u||\nabla u| \in L^{\frac{4}{3}}(\Omega).$$

The Lebegues Dominated Convergence Theorem implies

$$\lim_{n\to\infty}\int_\Omega (|\triangle u \cdot v^n|^2 + |(\triangle u \cdot v^n)u^n|^2 + |(\triangle u \cdot v^n)\nabla u|^{\frac{4}{3}})\,dx = 0$$

so that we have

$$\lim_{n\to\infty}\int_\Omega (\triangle u \cdot v^n)u^n \cdot \triangle v^n\,dx = 0,\ \lim_{n\to\infty}\int_\Omega (\triangle u \cdot v^n)\nabla u \cdot \nabla v^n\,dx = 0.$$

Now we need to show

$$\lim_{n\to\infty}\int_\Omega |\nabla u \cdot \nabla v^n|^2\,dx = 0. \tag{4.3}$$

Assume that (4.3) is true for the moment. Then it is easy to see $(\nabla v^n \cdot u^n)(\nabla u \cdot \nabla v^n) \to 0$ in $L^1(\Omega)$ so that we have

$$\int_\Omega |\nabla v^n|^4 = \int_\Omega |u^n \cdot \triangle v^n|^2\,dx + o(1) \le \int_\Omega |\triangle v^n|^2\,dx + o(1)$$

this clearly implies (4.2). To see (4.3), observe that $v^n \in W_0^{2,2}(\Omega, \mathbb{R}^5)$. Therefore we have by integration by parts,

$$\begin{aligned}
\int_\Omega |\nabla u \cdot \nabla v^n|^2\,dx &= \int_\Omega (\nabla u \cdot \nabla v^n)(\nabla u \cdot \nabla v^n)\,dx \\
&= -\int_\Omega \nabla \cdot ((\nabla u \cdot \nabla v^n)\nabla u) \cdot v^n\,dx \\
&= -2\int_\Omega (\triangle u \cdot \nabla v^n)\nabla u \cdot v^n\,dx - \int_\Omega (\nabla u \cdot \triangle v^n)(\nabla u \cdot v^n)\,dx \\
&\le C\int_\Omega (|\nabla v^n||\nabla u||\triangle u| + |\nabla u|^2|\triangle v^n|)\,dx \to 0, \text{ as } n \to \infty.
\end{aligned}$$

This gives (4.3).

Now we write

$$\int_\Omega D(u^n)\cdot\nabla\xi = A^n + B^n + C^n,$$

where

$$A^n = \int_\Omega u^n\cdot[(\frac{\partial u}{\partial x_2}\wedge\cdots\wedge\frac{\partial u}{\partial x_5})\frac{\partial\xi}{\partial x_1}+\cdots+(\frac{\partial u}{\partial x_1}\wedge\cdots\wedge\frac{\partial u}{\partial x_4})\frac{\partial\xi}{\partial x_5}]\,dx,$$

$$B^n = \int_\Omega [D(u^n)\cdot\nabla\xi\,dx - A^n - C^n$$

and

$$C^n = \int_\Omega V^n\cdot\nabla\xi\,dx,$$

where

$$V^n = (\det\,(u^n,\frac{\partial v^n}{\partial x_2},\cdots,\frac{\partial v^n}{\partial x_5}),\cdots,\ \det\,(u^n,\frac{\partial v^n}{\partial x_1},\cdots,\frac{\partial v^n}{\partial x_4})).$$

Since $u^n\to u$ weak* in $L^\infty(\Omega)$, we have

$$A^n\to\int_\Omega D(u)\cdot\nabla\xi\,dx,\ \text{as } n\to\infty.$$

To estimate B^n, we observe that direct calculations imply

$$\begin{aligned}
&\frac{\partial u^n}{\partial x_2}\wedge\frac{\partial u^n}{\partial x_3}\wedge\frac{\partial u^n}{\partial x_4}\wedge\frac{\partial u^n}{\partial x_5}-\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}\\
&=\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}+\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}\\
&+\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\frac{\partial u}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\frac{\partial u}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}\\
&+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\frac{\partial u}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}\\
&+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}\\
&+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial v^n}{\partial x_5}+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\frac{\partial v^n}{\partial x_2}\wedge\frac{\partial v^n}{\partial x_3}\wedge\frac{\partial v^n}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}
\end{aligned}$$

This implies

$$|B^n|\le C\int_\Omega(|\nabla u|^3|\nabla v^n|+|\nabla u|^2|\nabla v^n|^2+|\nabla u||\nabla v^n|^3)\,dx.$$

Since $\nabla v^n \to 0$ a.e. and weakly in $L^4(\Omega)$, we can conclude that $\nabla v^n \to 0$ strongly in $L^q(\Omega)$ for any $1 \le q < 4$. Therefore we see $B^n \to 0$ as $n \to \infty$.

Now we need to estimate C^n. Since $|u(x)| = 1$ for a.e. $x \in \Omega$, there exists a rotation $R \in SO(4)$ such that $R(u(x)) = (0,0,0,0,1)$. Moreover, for any vector $p_1, p_2, p_3, p_4, p_5) \in \mathbb{R}^5$, since

$$\det(u(x), p_{i_1}, \cdots, p_{i_4}) = \det(R(u(x)), R(p_{i_1}), \cdots, R(p_{i_4})), \forall 1 \le i_1 < i_2 < i_3 < i_4 \le 5.$$

We may assume that $u(x) = (0,0,0,0,1)$ and write $R(p_i) = (a_i, b_i, c_i, d_i, e_i)$ for $1 \le i \le 5$. Then we have

$$V = (\det(u(x), p_2, \cdots, p_5), \cdots, \det(u(x), p_1, \cdots, p_4)) = a \wedge b \wedge c \wedge d,$$

where $a = (a_1, \cdots, a_5), b = (b_1, \cdots, b_5), \cdots, d = (d_1, \cdots, d_5)$. Therefore we have

$$|V| = |a \wedge b \wedge c \wedge d| \le \frac{1}{2^4}(|a|^2 + |b|^2 + |c|^2 + |d|^2)^2. \tag{4.4}$$

Applying (4.4) with $p_i = \frac{\partial v^n}{\partial x_i}$ for $1 \le i \le 5$, we obtain

$$|V^n|(x) \le \frac{1}{16}|\nabla v^n|^4(x), \text{ for a.e. } x \in \Omega. \tag{4.5}$$

This, combined with (4.1) and (4.2), implies

$$\liminf_{n\to\infty}[\int_\Omega |\triangle u^n|^2 dx + 16\lambda \int_\Omega D(u^n) \cdot \nabla\xi\, dx] \ge \int_\Omega |\triangle u|^2 dx + 16\lambda \int_\Omega D(u) \cdot \nabla\xi\, dx.$$

This completes the proof of Lemma 4.1. □

As a direct consequence, we have

Corollary 4.2. *For any $\lambda \in (0,1]$, there exists a $u_\lambda \in W^{2,2}_\phi(\Omega, S^4)$ which minimizes $\mathbb{H}_\lambda(\cdot)$ over $W^{2,2}_\phi(\Omega, S^4)$.*

Proof. Since $L(u) \ge 0$ for any $u \in W^{2,2}_\phi(\Omega, S^4)$, it is easy to see that any minimizing sequence $\{u_i\}$ of $\mathbb{H}_\lambda(\cdot)$ over $W^{2,2}_\phi(\Omega, S^4)$ is a bounded sequence in $W^{2,2}(\Omega)$. Therefore we may assume that u_i converges to u_λ weakly in $W^{2,2}(\Omega)$. By Lemma 4.1, we have that u_λ is a minimizer for $\mathbb{H}_\lambda$ over $W^{2,2}_\phi(\Omega, S^4)$. □

Now we have

Lemma 4.3. *For any $\lambda \in (0,1)$, if $u_\lambda \in W^{2,2}_\phi(\Omega, S^4)$ is a minimizer for $\mathbb{H}_\lambda(\cdot)$. Then u_λ is a Q-minimizing biharmonic map, with $Q = \frac{1+\lambda}{1-\lambda}$.*

Proof. For simplicity, we abbreviate u_λ to u. Let $w \in W^{2,2}_\phi(\Omega, S^4)$. Then, similar to (4.5), we have

$$|D(w)|(x) \leq \frac{1}{16}|\nabla w|^4(x), \text{ for a.e. } x \in \Omega,$$

and

$$\begin{aligned} |L(w) - L(u)| &\leq \int_\Omega (|D(w)| + |D(u)|)\, dx \\ &\leq \frac{1}{16\sigma_4}\int_\Omega (|\nabla w|^4 + |\nabla u|^4)\, dx \leq \frac{1}{16\sigma_4}[\mathbb{H}(w) + \mathbb{H}(u)], \end{aligned} \tag{4.6}$$

where we have used the fact that $|\nabla w|^2 \leq |\bigtriangleup w|^2$ and $|\nabla u|^2 \leq |\bigtriangleup u|^2$ for a.e. $x \in \Omega$.

Since u minimizes $\mathbb{H}_\lambda$, we then have

$$\mathbb{H}(u) \leq \mathbb{H}(w) + 16\lambda\sigma_4(L(w) - L(u)) \leq \mathbb{H}(w) + \lambda(\mathbb{H}(w) + \mathbb{H}(u)).$$

This implies

$$\mathbb{H}(u) \leq \frac{1+\lambda}{1-\lambda}\mathbb{H}(w), \forall w \in W^{2,2}_\phi(\Omega, S^4). \tag{4.7}$$

Now we need to show that u_λ is a biharmonic map. To see it, let $\eta \in C^\infty_0(\Omega, \mathbb{R}^5)$, $t \in [0,1)$, and denote $u^t_\lambda(x) = \frac{u(x)+t\eta(x)}{|u(x)+t\eta(x)|}$ for $x \in \Omega$. Then we have

$$\frac{d}{dt}|_{t=0}(\mathbb{H}(u^t_\lambda) + 16\lambda\sigma_4 L(u^t_\lambda)) = 0. \tag{4.8}$$

Therefore u_λ is a biharmonic map, if we can show

$$\frac{d}{dt}|_{t=0}L(u^t_\lambda)) = 0. \tag{4.9}$$

In order to prove (4.9), we need the following Lemmas.

Lemma 4.4. *For any $u, v \in W^{2,2}_\phi(\Omega, S^4)$, we have the following inequality*

$$|L(u) - L(v)| \leq C\|\nabla(u-v)\|_{L^4(\Omega)}(\|\nabla u\|^3_{L^4(\Omega)} + \|\nabla v\|^3_{L^4(\Omega)}). \tag{4.10}$$

Proof. By the definition of L, we see

$$|L(u, u_0) - L(v, u_0)| \leq L(u,v) = \frac{1}{\sigma_4}\sup_{\xi:\Omega\to\mathbb{R}; |\nabla\xi|\leq 1}\int_\Omega (D(u) - D(v)) \cdot \nabla\xi\, dx \tag{4.11}$$

For any $\xi : \Omega \to R$ with $|\nabla \xi| \le 1$, we write

$$\int_\Omega (D(u) - D(v)) \cdot \nabla \xi \, dx = I + II + III + IV + V, \tag{4.12}$$

where

$$\begin{aligned} I = \int_\Omega [\det \left(u - v, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}, \frac{\partial u}{\partial x_5} \right) \frac{\partial \xi}{\partial x_1} + \cdots \\ + \det \left(u - v, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4} \right) \frac{\partial \xi}{\partial x_5}] \, dx, \end{aligned}$$

$$\begin{aligned} II = \int_\Omega [\det \left(v, \frac{\partial (u-v)}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}, \frac{\partial u}{\partial x_5} \right) \frac{\partial \xi}{\partial x_1} + \cdots \\ + \det \left(v, \frac{\partial (u-v)}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4} \right) \frac{\partial \xi}{\partial x_5}] \, dx, \end{aligned}$$

$$\begin{aligned} III = \int_\Omega [\det \left(v, \frac{\partial v}{\partial x_2}, \frac{\partial (u-v)}{\partial x_3}, \frac{\partial u}{\partial x_4}, \frac{\partial u}{\partial x_5} \right) \frac{\partial \xi}{\partial x_1} + \cdots \\ + \det \left(v, \frac{\partial v}{\partial x_1}, \frac{\partial (u-v)}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4} \right) \frac{\partial \xi}{\partial x_5}] \, dx, \end{aligned}$$

$$\begin{aligned} IV = \int_\Omega [\det \left(v, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3}, \frac{\partial (u-v)}{\partial x_4}, \frac{\partial u}{\partial x_5} \right) \frac{\partial \xi}{\partial x_1} + \cdots \\ + \det \left(v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial (u-v)}{\partial x_3}, \frac{\partial u}{\partial x_4} \right) \frac{\partial \xi}{\partial x_5}] \, dx, \end{aligned}$$

$$\begin{aligned} V = \int_\Omega [\det \left(v, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3}, \frac{\partial v}{\partial x_4}, \frac{\partial (u-v)}{\partial x_5} \right) \frac{\partial \xi}{\partial x_1} + \cdots \\ + \det \left(v, \frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3}, \frac{\partial (u-v)}{\partial x_4} \right) \frac{\partial \xi}{\partial x_5}] \, dx. \end{aligned}$$

It follows from Hölder's inequality that

$$|II| \le C \int_\Omega |\nabla (u-v)| |\nabla u|^3 \, dx,$$

$$\begin{aligned} |III| &\le C \int_\Omega |\nabla (u-v)| |\nabla u|^2 |\nabla v| \, dx \\ &\le C (\int_\Omega |\nabla (u-v)|^4 \, dx)^{1/4} (\int_\Omega (|\nabla u|^4 + |\nabla v|^4) \, dx)^{1/4}, \end{aligned}$$

$$
\begin{aligned}
|IV| &\le C\int_\Omega |\nabla(u-v)||\nabla u||\nabla v|^2\,dx\\
&\le C(\int_\Omega |\nabla(u-v)|^4\,dx)^{1/4}(\int_\Omega (|\nabla u|^4+|\nabla v|^4)\,dx)^{1/4},
\end{aligned}
$$

and

$$
|V| \le C\int_\Omega |\nabla(u-v)||\nabla v|^3\,dx \le C(\int_\Omega |\nabla(u-v)|^4\,dx)^{1/4}(\int_\Omega |\nabla v|^4\,dx)^{1/4}.
$$

In order to estimate I, we observe that

$$
\begin{aligned}
&4\left(\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}\right)\\
&=\left(u\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}\right)_{x_2}+\left(\frac{\partial u}{\partial x_2}\wedge u\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}\right)_{x_3}\\
&+\left(\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge u\wedge\frac{\partial u}{\partial x_5}\right)_{x_4}+\left(\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge u\right)_{x_5}
\end{aligned}
$$

in the sense of distributions. Therefore, by integration by parts, we have

$$
\begin{aligned}
&\int_\Omega (u-v)\cdot\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}\frac{\partial \xi}{\partial x_1}\\
&=\frac{1}{4}\int_\Omega [\frac{\partial(u-v)}{\partial x_2}\cdot u\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5}+\cdots\\
&+\frac{\partial(u-v)}{\partial x_5}\cdot\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge u]\frac{\partial \xi}{\partial x_1}\,dx\\
&+\frac{1}{4}\int_\Omega (u-v)\cdot[(u\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge\frac{\partial u}{\partial x_5})\frac{\partial^2 \xi}{\partial x_1\partial x_2}+\cdots\\
&+(\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_4}\wedge u)\frac{\partial^2 \xi}{\partial x_1\partial x_5}]\,dx.
\end{aligned}
$$

By doing the same calculations to all other terms in I, we see that the sum of all terms involving $\nabla^2\xi$ cancel each other. Therefore we have

$$
|I| \le C\int_\Omega |\nabla(u-v)|\nabla u|^3\,dx \le \left(\int_\Omega |\nabla(u-v)|^4\,dx\right)^{1/4}\left(\int_\Omega |\nabla u|^4\,dx\right)^{3/4}.
$$

Putting all these inequalities together, we obtain (4.10). $\square$

The next Lemma is concerning with the density of maps, which are smooth away from finitely many singular points, in $W^{2,2}_\phi(\Omega, S^4)$. The proof will be given in Section 5.

Lemma 4.5. *Define*

$R_\phi^\infty = \{u \in W_\phi^{2,2}(\Omega, S^4) : u \in C^\infty(\bar\Omega \setminus \cup_{i=1}^l \{a_i\}, S^4),$ *where* $l < \infty$ *and* $\cup_{i=1}^l \{a_i\} \subset \Omega\}.$

Then R_ϕ^∞ *is dense in* $W_\phi^{2,2}(\Omega, S^4)$ *for the* $W^{2,2}$*-topology.*

Now we return to the proof of Lemma 4.3. First we observe that for any $v \in R_\phi^\phi$ we have

$$L(\frac{v+t\eta}{|v+t\eta|}) = L(v), \text{ for sufficiently small } t \in [0,1)$$

since the singularity of $v^t = \frac{v+t\eta}{|v+t\eta|} \in R_\phi^\infty$ is same as that of v and $L(\cdot)$ is the minimal connection of its singular points ([BB] [BCL] [BBC] [GMS]).

For u_λ, it follows from Lemma 4.5 that there are $\{u_n\} \subset R_\phi^\infty$ such that

$$\lim_{n\to\infty} \|u_n - u_\lambda\|_{W^{2,2}(\Omega)} = 0.$$

Then, for sufficiently small $t \in [0,1)$, we also have

$$\lim_{n\to\infty} \|u_n^t - u_\lambda^t\|_{W^{2,2}(\Omega)} = 0$$

where $u_n^t = \frac{u_n+t\eta}{|u_n+t\eta|}$. By Lemma 4.4, we have

$$\lim_{n\to\infty} L(u_n^t) = L(u_\lambda^t), \ \lim_{n\to\infty} L(u_n) = L(u_\lambda).$$

On the other hand, since $u_n \in R_\phi^\infty$, we have, for any $t \in [0,1)$ sufficiently small,

$$L(u_n^t) = L(u_n).$$

Therefore we have $L(u_\lambda^t) = L(u)$ for any sufficiently small $t \in [0,1)$. This finishes the proof of Lemma 4.3. $\square$

We completion of proof of Theorem B.

Proof of theorem B. Part (a) and (b) follow from Lemma 4.1 and Corollary 4.2. Since Lemma 4.3 implies that each $\mathbb{H}_\lambda$-minimizer u_λ is a Q-minimizing biharmonic map with $Q = \frac{1+\lambda}{1-\lambda}$, part (c) follows from Corollary 3.2. $\square$

5. Appendix

In this section, we provide two examples, a proof of Lemma 4.5, a boundary partial regularity for $\mathbb{H}_\lambda$, and propose a few open questions.

Proposition A1. *For $n \geq 5$, $\Phi(x) = \frac{x}{|x|} : B^n \to S^{n-1}$ is a unique minimizing biharmonic map in $W^{2,2}_{\Phi}(B^5, S^4)$.*

Proof. Using the fact that $|u|^2 = 1$, we have

$$-u \cdot \triangle u = |\nabla u|^2 \tag{5.1}$$

Then

$$|\triangle u|^2 = |u \cdot \triangle u|^2 + |\triangle u \cdot \tau(u)|^2,$$

where $\tau(u) = (\tau_1, ..., \tau_{n-1})$ and $\{\tau_k(u)\}$ is an orthonormal base of the tangent plane of S^{n-1} at u. Since $\Phi : B^n \to S^{n-1}$ is a weakly harmonic map,

$$\triangle\Phi(x) \cdot \tau(\Phi(x)) = 0.$$

Now we recall that $\Phi : B^n \to S^{n-1}$ is a unique minimizing 4-harmonic map (cf. [CG], [AL], [Ho]). Then

$$\begin{aligned}\int_{\Omega} |\triangle u|^2 &= \int_{\Omega} |\nabla u|^4 + |\triangle u \cdot \tau(u)|^2 \, dx \\ &\geq \int_{\Omega} |\nabla \frac{x}{|x|}|^4 \, dx = \int_{\Omega} |\triangle \frac{x}{|x|}|^2 \, dx\end{aligned}$$

for all $u \in W^{2,2}_{\Phi}(B^n; S^{n-1})$. This implies that Φ is a unique minimizing biharmonic map. $\square$

Now we give an example consisting a domain $\Omega \subset \mathbb{R}^5$ and $\phi : \partial\Omega \to S^4$ with $\deg(\phi) = 0$ such that the infimum of $\mathbb{H}$ in $W^{2,2}_{\phi}(\Omega, S^4)$ is less than that in $C^{\infty}_{\phi}(\bar{\Omega}, S^4)$.

For a sufficiently large $L > 0$, let $B_1^+((0', L))$ (or $B_1^-((0', -L))$ resp.) be the upper (or lower, resp.) half unit ball centered at $(0', L)$ (or $(0', -L)$ resp.) in $\mathbb{R}^5$. Define

$$\Omega = B_1^+((0', L)) \cup (B_1^4 \times [-L, L]) \cup B_1^-((0', -L))$$

where $B_1^4 \subset \mathbb{R}^4$ is the unit ball centered at $0' \in \mathbb{R}^4$.

Let $\psi^+ : \partial B_1^+((0', L)) \cap \{x \in \mathbb{R}^5 : x_5 > L\} \to S^4$ be a smooth map of degree one such that

$$\psi^+|_{\partial B_1^+((0',L)) \cap \{x \in \mathbb{R}^5 : x_5 = L\}} = (0', 1), \ \frac{\partial \psi^+}{\partial x_5}|_{\partial B_1^+((0',L)) \cap \{x \in \mathbb{R}^5 : x_5 = L\}} = 0.$$

Define $\psi : \partial\Omega \to S^4$ by

$$\psi(x', x_5) = \begin{cases} \psi^+(x', x_5), \ x_5 \geq L \\ (0', 1) \qquad , \ x_5 \in [-L, L] \\ \psi^+(x', -x_5), \ x_5 \leq -L. \end{cases}$$

Then $\psi \in C^0(\partial\Omega, S^4) \cap W^{2,2}(\partial\Omega, S^4)$ has $\deg(\phi) = 0$. Motivated by the gap phenomena discovered by Hardt-Lin [HL] in the context of harmonic maps, we have

Proposition A2. *Under the above notations, we have the following gap phenomena*

$$\inf_{u \in W^{2,2}_{\psi}(\Omega, S^4)} \mathbb{H}(u) < \inf_{v \in W^{2,2}_{\psi}(\Omega, S^4) \cap C^0(\bar{\Omega}, S^4)} \mathbb{H}(v). \tag{5.5}$$

Proof. The idea is similar to that of [HL1]. First, observe that

$$\Psi(x) = \begin{cases} \psi(\frac{x-(0',L)}{|x-(0',L)|}), \quad x \in B_1^+((0', L)) \\ \psi(\frac{x-(0',-L)}{|x-(0',-L)|}), \quad x \in B_1^-((0', -L)) \\ (0', 1), \qquad\qquad x \in B_1^4 \times [-L, L]. \end{cases}$$

Then it is not difficult to verify that $\Psi \in W^{2,2}_{\psi}(\Omega, S^4)$. Moreover, direct calculations imply

$$\begin{aligned} \mathbb{H}(\Psi) &= 2 \int_{B_1^+((0',1))} |\triangle (\psi(\frac{x-(0',L)}{|x-(0',L)|}))|^2 \, dx \\ &= 2 \int_0^1 dr \int_{\partial B_1^+(0',L)} |\triangle \psi^+|^2 = C(\psi^+) \end{aligned}$$

is independent of L.

On the other hand, for any $v \in W^{2,2}_{\psi}(\Omega, S^4) \cap C(\bar{\Omega}, S^4)$, since $|\triangle v|(x) \geq |\nabla v|^2(x)$ for a.e. $x \in \Omega$, we have

(5.6)

$$\begin{aligned} &\mathbb{H}(v) \geq \int_{B_1^4 \times [-L,L]} |\triangle v|^2 \, dx \geq \int_{-L}^{L} dx_5 \int_{B_1^4} |\triangle v|^2 \, dx' \\ &\geq \int_{-L}^{L} dx_5 \int_{B_1^4} |\nabla_{x'} v|^4(x', x_5) \, dx' \geq 16 \int_{-L}^{L} dx_5 \int_{B_1^4} |\det(\nabla_{x'} v|(x', x_5) \, dx' \\ &\geq 16\sigma_4 \int_{-L}^{L} dx_5 = 32\sigma_4 L \end{aligned}$$

where we have used the inequality (4.5) and

$$\int_{B_1^4} |\det(\nabla_{x'} v)|(x', x_5)\, dx' \geq \sigma_4,\ \forall x_5 \in (-L, L). \tag{5.7}$$

(5.7) holds, since for any $x_5 \in (-L, L)$ $v \in C(\overline{B_1^+(0', L) \cup B_1^4 \times [x_5, L]}, S^4)$ hence $v : \partial(B_1^+(0', L) \cup B_1^4 \times [x_5, L]) \to S^4$ has degree zero.

In particular, we have that $v(\cdot, x_5) : B_1^4 \to S^4$ has degree one for all $x_5 \in (-L, L)$.

Therefore we establish (5.5), provided that $L > 0$ is chosen to be sufficiently large. $\square$

Now,we complete the proof of Lemma 4.5.

Proof of Lemma 4.5. The idea is similar to that of [BZ]. Since $\phi \in C^\infty(\bar{\Omega} \setminus \{x_i\}_{i=1}^k, S^4)$ for some $\{x_i\}_{i=1}^k \subset \Omega$, we have that for any $u \in W^{2,2}_\phi(\Omega, S^4)$ there are $\{u_n\} \subset C^\infty(\bar{\Omega}, \mathbb{R}^5)$ such that $u_n = \phi$, $\nabla u_n = \nabla\phi$ on $\partial\Omega$, and $u_n \to u$ strongly in $W^{2,2}(\Omega, S^4)$. For any small $\epsilon > 0$, set

$$S^4_{1-\varepsilon} = \{x \in \mathbb{R}^5 : |x| = 1 - \varepsilon\}, \quad S^4_{1+\varepsilon} = \{x \in \mathbb{R}^5 : |x| = 1 + \varepsilon\}.$$

By the Sard's theorem, we have that

$$F^-_{n,\varepsilon} = u_n^{-1}(S^4_{1-\varepsilon}), \quad F^+_{n,\varepsilon} = u_n^{-1}(S^4_{1+\varepsilon})$$

are two compact submanifolds of Ω of codimension one. Moreover

$$V^-_{n,\varepsilon} = u_n^{-1}(\{|y| \leq 1 - \varepsilon\}), \quad V^+_{n,\varepsilon} = u_n^{-1}(\{|y| \geq 1 + \varepsilon\})$$

are smooth domains inside Ω such that $\partial V^-_{n,\epsilon} = F^-_{n,\varepsilon}$ and $\partial V^+_{n,\epsilon} = F^+_{n,\varepsilon}$.

For any $a \in B^5_{\frac{1}{2}}$, define the projection maps $\Pi^-_a : \mathbb{R}^5 \to S^4_{1-\varepsilon}$ and $\Pi^+_a : \mathbb{R}^5 \to S^4_{1+\varepsilon}$ by

$$\Pi^-_a(x) = \frac{x - a}{|x - a|}(1 - \varepsilon), \quad \Pi^+_a(x) = \frac{x - a}{|x - a|}(1 + \varepsilon).$$

By Lemma 2.1, there exist $a_1, a_2 \in B^5_{\frac{1}{2}}$ such that the maps $h^-_{n,\varepsilon} : V^-_{n,\epsilon} \to S^4_{1-\epsilon}$, $h^+_{n,\varepsilon} : V^+_{n,\epsilon} \to S^4_{1+\epsilon}$ defined by

$$h^-_{n,\varepsilon} = (\Pi^-_{a_1}|_{S^4_{1-\varepsilon}})^{-1} \circ \Pi^-_{a_1} \circ u_n, \quad h^+_{n,\varepsilon} = (\Pi^+_{a_2}|_{S^4_{1+\varepsilon}})^{-1} \circ \Pi^+_{a_2} \circ u_n$$

satisfy

$$h^-_{n,\varepsilon} - u_n \in W^{2,2}_0(V^-_{n,\varepsilon}, \mathbb{R}^5), \quad h^+_{n,\varepsilon} - u_n \in W^{2,2}_0(V^+_{n,\varepsilon}, \mathbb{R}^5),$$

$$\int_{V^-_{n,\varepsilon}} |\nabla^2 h^-_{n,\varepsilon}|^2\, dx \leq C \int_{V^-_{n,\varepsilon}} (|\nabla^2 u_n|^2 + |\nabla u_n|^4)\, dx$$

and

$$\int_{V^+_{n,\varepsilon}} |\nabla^2 h^+_{n,\varepsilon}|^2\, dx \le C \int_{V^+_{n,\varepsilon}} (|\nabla^2 u_n|^2 + |\nabla u_n|^4)\, dx.$$

We now define

$$w_{n,\varepsilon}(x) = \begin{cases} h^+_n(x) & \text{for } x \in V^+_{n,\varepsilon} \\ h^-_n(x) & \text{for } x \in V^-_{n,\varepsilon} \\ u_n(x) & \text{for } x \notin V^+_{n,\varepsilon} \cup V^-_{n,\varepsilon}. \end{cases}$$

Then it is easy to see that $w_{n,\varepsilon} \in W^{2,2}_\phi(B^5, S^4)$ has only a finitely many singular points in Ω and satisfies

$$\int_{V^+_{n,\epsilon} \cup V^-_{n,\epsilon}} |\nabla^2 w_{n,\epsilon}|^2\, dx \le C \int_{V^+_{n,\varepsilon} \cup V^-_{n,\varepsilon}} |\nabla^2 u_n|^2 + |\nabla u_n|^4)\, dx \to 0, \quad \text{as } n \to \infty$$

since $u_n \to u$ strongly in $W^{2,2}$ and $\lim_{n\to\infty} |V^+_{n,\epsilon} \cup V^-_{n,\epsilon}| = 0$.

Finally, to obtain the desired approximation, we only have to project $w_{n,\epsilon}$ on S^4 and let $\varepsilon \to 0$. $\square$

Proposition A3. *Let $\Phi(x) = \frac{x}{|x|} : B^5 \to S^4$. Then there exist infinitely many biharmonic maps $\{u_i\} \subset W^{2,2}_\Phi(B^5, S^4)$, each of which is smooth away from a closed set Σ_i with $\mathcal{H}^{1-\delta}(\Sigma_i) = 0$ for some $\delta > 0$.*

Proof. It is based on some modifications of [BBC]. First, let $u_0 \ne \Phi$ be a given map in $W^{2,2}_\Phi(B^5, S^4)$ having finitely many interior singular points. For $0 < \lambda < 1$, let $u_\lambda \in W^{2,2}_\Phi(B^5, S^4)$ be a minimizer in $W^{2,2}_\Phi(B^5, S^4)$ of

$$\hat{\mathbb{H}}_\lambda(v) := \mathbb{H}(v) + 16\sigma_4 \lambda L(u, u_0)$$

where

$$L(u, u_0) = \frac{1}{\sigma_4} \sup_{\xi:\Omega\to\mathbb{R}, \|\nabla\xi\|_{L^\infty} \le 1} \int_\Omega (D(u)\cdot\nabla\xi - D(u_0)\cdot\nabla\xi)\, dx.$$

We remark that theorem B, corollary 3.2, and Lemma 4.3 also hold for minimizers of $\hat{\mathbb{H}}_\lambda$.

We claim: for $0 < \lambda < 1$, $u_\lambda \ne \Phi$. For, otherwise, Φ is a minimizer for both $\mathbb{H}$ and $\hat{\mathbb{H}}_\lambda$. In particular, setting $\Phi(t) = \Phi \circ \eta(t)$, we have

$$\left.\frac{d}{dt} L(\Phi(t), u_0)\right|_{t=0} = 0 \tag{5.8}$$

where $\eta(t)$ is a smooth family of diffeomorphisms from B^5 into itself, satisfying $\eta(0) = Id$ and $\eta(t) = Id$ on ∂B. This is impossible, for we can choose suitable $\eta(t)$ such that $L(\Phi(t), u_0) = L(\Phi, u_0) - t + o(t)$ as $t \to 0$.

For a fixed $\lambda_1 \in (0,1)$, let

$$A_1 = \min\{\mathbb{H}(v) : \ v \text{ is a minimizer of } \hat{\mathbb{H}}_{\lambda_1} \}.$$

Then there exists a map $u_{\lambda_1} \in W^{2,2}_{\Phi}(B^5, S^4)$ which minimizes $\hat{\mathbb{H}}_{\lambda_1}$ such that $\mathbb{H}(u_{\lambda_1}) = A_1$. Moreover, since

$$\mathbb{H}(\Phi) < A_1$$

there exists a sufficiently small $0 < \lambda_2 < \lambda_1$ such that

$$\mathbb{H}(\Phi) + \lambda_2[\mathbb{H}(u_0) + \mathbb{H}(\Phi)] < A_1.$$

Let $u_{\lambda_2} \in W^{2,2}_{\Phi}(B^5, S^4)$ be a minimizer of $\hat{\mathbb{H}}_{\lambda_2}$. Then we have

$$\begin{aligned}
&\mathbb{H}(u_{\lambda_2}) \le \hat{\mathbb{H}}_{\lambda_2}(u_{\lambda_2}) \le \hat{\mathbb{H}}_{\lambda_2}(\Phi) \\
&\le \mathbb{H}(\Phi) + \lambda_2 \int_{B^5} [|\nabla \Phi|^4 + |\nabla u_0|^4]\, dx \le \mathbb{H}(\Phi) + \lambda_2[\mathbb{H}(\Phi) + \mathbb{H}(u_0)] < A_1.
\end{aligned}$$

This implies that u_{λ_2} is different from both Φ and u_{λ_1} Iterating this construction, we find infinitely many biharmonic maps u_{λ_l}. By Theorem 3.2 and Lemma 4.3, each u_{λ_l} is partially regular. This proves Proposition A3. □

References

[AL] M. Avellenada and F. H. Lin, *Fonctions quasi affines et minimization de* $\int |\nabla u|^p$, C. R. Acad. Sci. Paris **306** (1988), 355–358.

[BB] F. Bethuel and H. Brezis, *Regularity of minimizers of relaxed problems for harmonic maps*, J. Funct. Anal. **101** (1991), 145–161.

[BBC] F. Bethuel, H. Brezis and J. M. Coron, *Relaxed energies for harmonic maps*, In variational methods, edited by Berestycki, Coron, Ekeland, Birkhäuser, Basel (1990), 37–52.

[BCL] H. Brezis, J.-M. Coron and E.H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649–705.

[BZ] F. Bethuel and X. Zheng, *Density of smooth functions between two manifols in Sobolev spaces*, J. Funct. Anal. **80** (1988), 60–75.

[CG] J. M. Coron and R. Gulliver, *Minimizing p-harmonic maps into spheres*, J. Reine Angew. Math. **401** (1989), 82–100..

[CWY] S. Y. A. Chang, L. Wang and P. Yang, *A regularity theory of biharmonic maps*, Comm. Pure Appl. Math. **52** (1999), 1113–1137.

[F] H. Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc.vol 76 (1970), 767–771.

[G] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton Univ. Press, 1983.

[GG] M. Giaquinta and E. Giusti, *The singular set of the minima of certain quadratic functionals*, Ann. Scuola Norm. Sup. Pisa (4) **11** (1984), 45–55.

[GMS] M. Giaquinta, G. Modica and J. Soucek, *The Dirichlet energy of mappings with values into the sphere*, Manuscripta Math. **65** (1989), 489–507.

[HKL] R. Hardt, D. Kinderlehrer and F.-H. Lin, *Stable defects of minimizers of constrained variational principles*, Ann. Inst. Henri Poincaré, Analyse non linéaire **5** (1988), 297–322.

[HL] R. Hardt, F. H. Lin, *Mappings minimizing the L^p norm of the gradient*, Comm. Pure Appl. Math. **40** (1987), 555–588.

[HL1] R. Hardt and F.-H. Lin, *A remark on H^1 mappings*, Manus. Math. **56** (1986), 1–10.

[HM] R. Hardt and L. Mou, *Minimizing biharmonic maps from $\mathbb{R}^4$ to spheres*, unpublished notes.

[Ho] M.-C. Hong, *On the minimality of the p-harmonic map $\frac{x}{|x|} : B^n \to S^{n-1}$*, Calc. Var. & PDES **13** (2001), 459–468.

[L] F. H. Lin, *Une remarque sur L'application $x/|x|$*, C. R. Acad. Sc. Paris **305** (1987), 529–531.

[LW] T. Lamm and C. Y. Wang, *Boundary regularity for biharmonic maps*, In preparation.

[M] C. B. Morrey, *Multiple integrals in the calculus of variations.*, Springer-Verlag New York, Inc., New York (1966).

[SU] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Diff. Geom. **17** (1982), 307–335.

[S] L. Simon, *Theorems on regularity and singularity of energy minmizing maps*, Lecture note in Mathematics ETH Zürich, Birkhäuser verlage, Basel, 1996.

[W1] C. Y. Wang, *Remarks on biharmonic maps into spheres*, Calc. Var. & PDE (to appear).

[W2] C. Y. Wang, *Biharmonic maps form $\mathbb{R}^4$ into a Riemannian manifold*, Math. Z. (to appear).

[W3] C. Y. Wang, *Stationary biharmonic maps form $\mathbb{R}^m$ into a Riemannian manifold*, Comm. Pure Appl. Math. **LVII** (2004), 0419-0444.

Min-Chun Hong, Department of Mathematics, University of Queensland, Brisbane, QLD 4072 , Australia

Changyou Wang, Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA